

# On a Gibbs characterization of normalized generalized Gamma processes.<sup>\*†</sup>

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## Abstract

We show that a Gibbs characterization of normalized generalized Gamma processes, recently obtained in Lijoi, Prünster and Walker (2007), can alternatively be derived by exploiting a characterization of exponentially tilted Poisson-Kingman models stated in Pitman (2003). We also provide a completion of this result investigating the existence of normalized random measures inducing exchangeable Gibbs partitions of type  $\alpha \in (-\infty, 0]$ .

*Keywords:* Exchangeable random partitions; Exponential tilting; Generalized Gamma processes; Gibbs partitions; Normalized random measures.

## 1 Introduction

The idea of constructing random probability measures by normalizing the increments of a random process dates back to Kingman's construction of the Dirichlet process by normalization of the increments of a Gamma process (Kingman, 1975). Since then it has been exploited in a variety of contexts, like e.g. ecology (Engen, 1978), population genetics (Ewens and Tavaré, 1995), excursion theory (Pitman and Yor, 1997) and combinatorics (Pitman, 2006). Recently the normalization approach has gained new interest in Bayesian nonparametrics, where a key problem is to define tractable alternatives to the Dirichlet process to be used as priors on the space of probability distributions (see e.g. James, 2002; Regazzini, Lijoi and Prünster, 2003). A comprehensive Bayesian analysis of a large class of random probability measures obtained by an appropriate normalization is proposed in James, Lijoi and Prünster (2005) by providing explicit marginal distributions, tractable analogues of the Blackwell-MacQueen Pólya urn prediction rules, and suitable descriptions of posterior distributions.

Normalized random measures select almost surely *discrete* probability measures (see e.g. James, 2003), which are well-known to induce exchangeable random partitions of the positive integers (Kingman, 1978). Among those the subset characterized by the so-called *Gibbs* product form of the partition probability function, has been recently shown in Gneden and Pitman (2006) to be a convex set, whose extreme points, for different values of the *type* parameter  $\alpha \in (-\infty, 1)$ , have been derived by resorting to the theory of Stirling triangles. In Bayesian nonparametrics the product form of the exchangeable partition probability function (EPPF) is highly desirable both with regard to mathematical tractability of prior-to-posterior analysis, like in connection with sequential

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sampling schemes for posterior's simulation, hence it is of some interest to characterize the subclass of normalized random measures which possesses this property.

Lijoi, Prünster and Walker (2007) have recently shown by means of an analytical approach based on solutions of a specific recursive equation that, in the space of (homogeneous) normalized random measures, the Gibbs product form of the EPPF for  $\alpha \in (0, 1)$  characterizes the class of normalized *generalized Gamma processes*, i.e. of random discrete probability measures whose ranked atoms follow the *exponentially tilted Poisson-Kingman distribution* derived from the positive  $\alpha$ -stable subordinator (Pitman, 2003; see also Cerquetti, 2007). Here we show such a characterization can also be obtained relying on known results on Poisson-Kingman models. In particular it arises by exploiting a characterization of exponentially tilted Poisson-Kingman models stated without proof in Pitman (2003). We also complete the study of normalized random measures inducing exchangeable partitions in Gibbs product form by discussing the cases  $\alpha = 0$  and  $\alpha < 0$  which have not been treated in Lijoi, Prünster and Walker (2007).

The paper is organized as follows. In Section 2 we recall basic definitions and results on normalized random measures, exchangeable random partitions and Poisson-Kingman models. In Section 3 we recall Gneden and Pitman's (2006) characterization of Gibbs partitions in terms of extremes points and propose a detailed discussion of this fundamental result. Finally, in Section 4, after providing a proof of Pitman's characterization of exponentially tilted Poisson-Kingman models, we establish our main result.

## 2 Preliminaries and basic definitions

We start by providing a very general and well-known construction of *homogeneous* normalized random measures. First recall that given a strictly positive random variable  $T$ , with density  $f_T$  and Laplace transform

$$E(e^{-\lambda T}) = \int_0^\infty e^{-\lambda t} f_T(t) dt = e^{-\psi(\lambda)}$$

where, according to the Lévy-Kintchine formula, for  $\lambda > 0$ ,  $\psi(\lambda) = \int_0^\infty (1 - e^{-\lambda x}) \rho(x) dx$  is the Laplace exponent, (i.e.  $T$  is an infinite divisible r.v.),  $f_T(\cdot)$  is uniquely identified by its unique Lévy density  $\rho(\cdot)$ , which satisfies  $\int_0^\infty \rho(x) dx = \infty$ , (otherwise  $P(T = 0) = \exp(-\psi(\infty)) > 0$  contrary to the previous assumption that  $T$  is strictly positive). Let  $H(\cdot)$  denote a probability measure on a Polish space  $(S, \mathcal{S})$ , fixed and non-atomic. Now, for each  $T$  and  $H$ , one may construct (Kingman, 1967) a *completely random measure*  $\mu$  on  $\mathcal{S}$ , characterized by its Laplace functional for every positive measurable function  $g$  on  $S$  as

$$\mathbb{E}[e^{-\mu(g)} | H] = \exp \left\{ - \int_S \psi(g(s)) H(ds) \right\} = \exp \left\{ - \int_S \int_0^\infty (1 - e^{-g(s)x}) \rho(dx) H(ds) \right\}$$

where  $\mu(g) = \int_S g(s) \mu(ds)$ , so that  $T = \mu(S) := \int_S \mu(ds) = \int_S I\{s \in S\} \mu(ds)$ . An (*homogeneous*) *normalized random measure* (NRM)  $P(\cdot)$  on  $(S, \mathcal{S})$  is then obtained by normalizing  $\mu$  as follows

$$P(\cdot) := \frac{\mu(\cdot)}{\mu(S)} = \frac{\mu(\cdot)}{T}.$$

Notice that a more general construction, incorporating *non-homogeneous* NRMs based on non-homogeneous Lévy measures, can be given in terms of mean intensity  $\nu(dx, ds) = \rho(dx|s)H(ds)$  of a Poisson random measure  $N(dx, ds)$  on  $(\mathcal{X} \times \mathcal{S})$  characterized by Laplace functional

$$\mathbb{E}[e^{-N(g)}] = \exp \left\{ - \int_S \int_0^\infty (1 - e^{-g(s,x)}) \rho(dx|s) H(ds) \right\},$$

(see e.g. James, 2002; James, 2005; James, Lijoi and Prünster, 2005), but here we do not need to deal with such a general construction.

As proved e.g. in James (2003) NRMs select almost surely discrete distributions, and it is well known that given a law  $Q$  on the space  $\mathcal{P}_1^\downarrow$  of decreasing sequences of positive numbers with sum 1, and a law  $H(\cdot)$  on a Polish space  $(S, \mathcal{S})$ , a *random discrete* probability measure (RDPM)  $P$  on  $\mathcal{S}$  may always be defined as  $P(\cdot) = \sum_{i=1}^\infty P_i \delta_{X_i}(\cdot)$ , for  $X_i$  iid  $\sim H(\cdot)$  and  $(P_i) \sim Q$ . From Kingman's theory of exchangeable random partitions (Kingman, 1978), sampling from  $P$  induces a random partition  $\Pi$  of the positive integers  $\mathbb{N}$  by the exchangeable equivalence relation  $i \approx j \Leftrightarrow X_i = X_j$ , that is to say two positive integers  $i$  and  $j$  belong to the same block of  $\Pi$  if and only if  $X_i = X_j$ , where  $X_i|P$  are iid  $\sim P$ . It follows that, for each restriction  $\Pi_n = \{A_1, \dots, A_k\}$  of  $\Pi$  to  $[n] = \{1, \dots, n\}$ , and for each  $n = 1, 2, \dots$ ,

$$Pr(\Pi_n = \{A_1, \dots, A_k\}) = p(n_1, \dots, n_k),$$

where, for  $j = 1, 2, \dots, k$ ,  $n_j = |A_j| \geq 1$  and  $\sum_{j=1}^k n_j = n$ , for some non-negative symmetric function  $p$  of finite sequences of positive integers called the *exchangeable partition probability function* (EPPF) determined by  $\Pi$  (see Pitman, 2006, for a comprehensive account on exchangeable random partitions and related stochastic processes).

Pitman (2003), generalizing Kingman's (1975) construction of the Dirichlet process as a Gamma process with independent increments divided by the sum, introduces a large class of RDPMs deriving the law  $Q$  by a random discrete distribution  $(P_i) = (J_i/T)$ , where  $J_1 \geq J_2 \geq \dots \geq 0$  are the random lengths of the ranked points of a Poisson process with Lévy density  $\rho$  and  $T = \sum_i J_i$ . It is easy to see that this construction is formally equivalent to the homogeneous normalized random measure's construction given above, so that, for  $(X_i)$ , independent of  $(P_i)$ , iid  $\sim H$ ,

$$P(\cdot) = \frac{\mu(\cdot)}{T} = \sum_{i=1}^\infty \frac{J_i}{T} \delta_{X_i}(\cdot).$$

Pitman termed the laws  $Q$  of  $(P_i)$  on  $\mathcal{P}_1^\downarrow$  *Poisson-Kingman distributions with Lévy density  $\rho$* , and also enlarged the *basic* Poisson-Kingman family by considering the larger class of *Poisson-Kingman distributions with Lévy density  $\rho$  and mixing distribution  $\gamma$*  given by

$$PK(\rho, \gamma) := \int_0^\infty PK(\rho|t) \gamma(dt), \quad (1)$$

where  $PK(\rho|t)$  is the regular conditional distribution of  $(P_i)$  given  $(T = t)$  constructed above, and  $\gamma$  is an arbitrary probability distribution on  $(0, \infty)$ . Clearly if  $\gamma(\cdot) = f_T(\cdot)$  then  $PK(\rho, \gamma) = PK(\rho)$ .

To deal with what follows it is also worthwhile to recall that, for  $(\tilde{P}_1, \tilde{P}_2, \dots)$  a *size-biased permutation* of the ranked atoms  $(P_i)$  of  $P(\cdot)$ , and  $\tilde{\nu}(dp)$  the distribution of  $\tilde{P}_1$  on  $(0, 1]$  (also termed the *structural distribution* of  $P(\cdot)$  in Engen, 1978), from Theorem 2.1 in Perman, Pitman and Yor (1992), (see also Lemma 1 and 2 in Pitman, 2003), the conditional distribution  $PK(\rho|t)$  for  $(P_i)|T = t$ , is completely described by means of the density of the conditional *structural distribution* of  $\tilde{P}_1|T = t$  given by

$$\tilde{f}(p|t) := pt\rho(pt) \frac{f(\bar{p}t)}{f(t)} \quad (2)$$

for  $0 < p < 1$ ,  $\bar{p} := 1 - p$ ,  $f(\cdot)$  the probability density of  $T$  and  $\rho(\cdot)$  the corresponding Lévy density. This implies that the density  $\tilde{f}(p|t)$ , joint with the density  $\gamma(t)$ , give a complete description of a mixed  $PK(\rho, \gamma)$  model.

**Example 1.** [Dirichlet process] For  $\rho_\theta(x) = \theta x^{-1}e^{-x}$ ,  $\theta > 0$ , the Lévy density of the Gamma  $(\theta, 1)$  density, with Laplace exponent  $\psi(\lambda) = \theta \log(1 + \lambda)$ , the law  $PK(\rho_\theta)$  gives the Poisson-Dirichlet  $(\theta)$  distribution governing the ranked atoms of the *normalized Gamma process*,  $P(\cdot) = \mu(\cdot)/T$ , for  $T \sim \text{Gamma}(\theta, 1)$ , and  $\mu(\cdot) \sim \text{Gamma}(\theta H(\cdot), 1)$ , which is well-known to correspond to the Dirichlet process with parameter measure  $\theta H(\cdot)$ , (Ferguson, 1973; Kingman, 1975). By (2) it is easy to check that structural conditional and unconditional distributions coincide and are both Beta  $(1, \theta)$  with density  $\tilde{f}(p) = \theta(1-p)^{\theta-1}$ , for  $0 < p < 1$ . This implies  $PK(\rho_\theta) = PK(\rho_\theta|t)$  for every  $t$ , and  $PK(\rho_\theta) = PK(\rho_\theta, \gamma)$ , for every  $\gamma$ . (See Pitman, 1996)

### 3 Normalized random measures inducing Gibbs partitions

By Definition 1 in Gneden and Pitman (2006), an exchangeable random partition  $\Pi$  of the positive integers, is said to be of *Gibbs form* if for some nonnegative weights  $W = (W_j)$  and  $V = (V_{n,k})$  the EPPF of  $\Pi$  can be expressed in the product form

$$p(n_1, \dots, n_k) = V_{n,k} \prod_{j=1}^k W_{n_j}$$

for all  $1 \leq k \leq n$ , and all compositions  $(n_1, \dots, n_k)$  of  $n$ . Gneden and Pitman (2006) also show that to define an *infinite* random partition of  $\mathbb{N}$ , i.e. a sequence  $(\Pi_n)$  consistent as  $n$  varies, the weights  $(W_j)$  must be of the following very special form depending on a single parameter  $\alpha \in [-\infty, 1)$ ,

$$W_{n_j} = (1 - \alpha)_{n_j-1\uparrow}$$

(with  $W_j = 1$  for every  $j$ , for  $\alpha = -\infty$ ), and the weights  $(V_{n,k})$  must be the solution to the backward recursion

$$V_{n,k} = (n - \alpha k)V_{n+1,k} + V_{n+1,k+1}$$

with  $V_{1,1} = 1$ . The solutions, for each  $\alpha < 1$ , are then obtained identifying the extreme points of the infinite dimensional simplex of the possible weights  $V$ , and deriving corresponding families of extreme partitions, in terms of the laws of the corresponding ranked atoms  $(P_i)$ , by a combinatorial technique based on the theory of Stirling triangles. The fundamental result, already stated without proof in Pitman (2003, cfr. Th. 8), is the following:

**Theorem 1.** [Gneden and Pitman, 2006; Th. 12] Each exchangeable Gibbs partition of a fixed type  $\alpha \in [-\infty, 1)$ , i.e. characterized by an EPPF of the form

$$p(n_1, \dots, n_k) = V_{n,k} \prod_{j=1}^k (1 - \alpha)_{n_j-1\uparrow} \tag{3}$$

is a unique probability mixture of extreme partitions of this type, which are

- a) for  $\alpha \in [-\infty, 0)$   $PD(\alpha, m|\alpha|)$  partitions with  $m = 0, 1, \dots, \infty$ ,
- b) for  $\alpha = 0$   $PD(0, \theta)$  partitions with  $\theta \in [0, \infty)$ ,
- c) for  $\alpha \in (0, 1)$   $PK(\rho_\alpha|t)$  partitions with  $t \in [0, \infty)$ .

Recall that for  $0 \leq \alpha < 1$  and  $\theta > -\alpha$ , or  $\alpha < 0$  and  $\theta = m|\alpha|$ ,  $PD(\alpha, \theta)$  stands for the two-parameter extension of the Poisson-Dirichlet distribution (Pitman and Yor, 1997), whose general form of the EPPF is well-known to be

$$p_{\alpha, \theta}(n_1, \dots, n_k) = \frac{(\theta + \alpha)_{k-1; \alpha \uparrow}}{(1 + \theta)_{n-1 \uparrow}} \prod_{i=1}^k (1 - \alpha)_{n_i - 1 \uparrow}. \quad (4)$$

(where  $k-1; \alpha \uparrow$  stands for the usual notation of rising factorial), and that, for  $\alpha \in (0, 1)$ ,  $\rho_\alpha(\cdot)$  is the Lévy density of the positive  $\alpha$ -stable distribution of index  $\alpha \in (0, 1)$ . The EPPF induced by the conditional model  $PK(\rho_\alpha|t)$  has been derived in Pitman (2003, Eq. (66)) and is given by

$$p_\alpha(n_1, \dots, n_k|t) = \frac{\Gamma(1 - \alpha)}{\Gamma(n - k\alpha)} \left(\frac{\alpha}{t}\right)^{k-1} \mu_\alpha(n - 1 - k\alpha + \alpha|t) \prod_{i=1}^k (1 - \alpha)_{n_i - 1 \uparrow}$$

for  $\mu_\alpha(q|t) = \mathbb{E}_\alpha(\tilde{P}_1^q|t)$ .

Relying on Theorem 1 the problem to identify NRMs inducing EPPF of Gibbs product form (3), reduces to identify mixing distributions  $\gamma$ , respectively on  $\mathbb{N} \cup \{0\}$  for case a), and on  $[0, \infty)$  for case b) and c), such that the resulting partition mixture model define the ranked atoms of RDPMS which can be obtained by normalizing a completely random measures. In other words we look for mixing distributions  $\gamma(\cdot)$  yielding corresponding strictly positive infinite divisible random variables  $T$  on which to build, as from Section 2, corresponding completely random measures. Notice that in Lijoi, Prünster and Walker (2007) the discussion is confined to case c), i.e. they characterize the class of *normalized Generalized Gamma processes* (Pitman, 2003) as being the unique family of normalized random measures inducing Gibbs partitions of type  $\alpha \in (0, 1)$ . Here we complete the discussion by investigating also cases a) and b). Moreover we show that the characterization of case c) arises as a direct consequence of a characterization of *exponentially tilted* Poisson-Kingman models stated in Pitman (2003) for which we also provide a proof.

**Case a)** For  $\alpha \in [-\infty, 0)$  a law  $Q$  on  $\mathcal{P}_1^\perp$  induces an EPPF of Gibbs form (3) if

$$Q_{\alpha, \gamma}(\cdot) = \sum_{m=0}^{\infty} PD(\alpha, m|\alpha|) \gamma(m)$$

for  $\gamma(\cdot)$  a probability distribution on the space of non negative integers. Recall that, for each  $m$ ,  $PD(\alpha, m|\alpha|)$  are Fisher's models for species sampling (Fisher et al. 1943, see also Pitman, 1996) and correspond to symmetric Dirichlet random vectors of dimension  $m$  and parameter  $|\alpha|$ . This implies that a random  $P$  whose ranked atoms are  $PD(\alpha, m|\alpha|)$  distributed can be constructed as

$$P(\cdot) = \sum_{i=0}^m \frac{G_i}{G} \delta_{X_i}(\cdot)$$

where  $G_i$  are iid Gamma  $(-\alpha, 1)$  r.v.'s,  $G = \sum_{i=1}^m G_i$  is Gamma  $(-\alpha m, 1)$ , and  $(X_i)$  are iid  $\sim H$  independent of  $(G_i)$ . In Bayesian nonparametric literature  $P(\cdot)$  is also called a *finite dimensional Dirichlet prior* (see e.g. Ishwaran and Zarepour, 2002). Mixing with respect to  $m$  corresponds to randomize the number  $m$  of terms in  $G$ , so that  $G = \sum_{i=0}^M G_i$ , for  $M \sim \gamma(\cdot)$ , has *compound  $\gamma$ -Gamma distributions*, with a positive mass at zero. To our purpose we want  $G$  to be infinite divisible (ID) with Lévy measure having infinite total mass. From e.g. Theorem 3.2 in Steutel and Van Harn (2004) a compound distribution with positive mass at zero is ID if and only if it is a

*compound Poisson*, hence we need to restrict our search to  $\gamma$  having Poisson distribution. Recall that compound Poisson-Gamma distributions (Aalen, 1992) belong to the family of *generalized Gamma* distributions (see e.g. Brix 1999), a class of infinite divisible distributions defined for  $\alpha < 1$  and characterized by Laplace exponent of the form

$$\psi_{GG}(\lambda) = -\delta[\zeta + (\zeta^{\frac{1}{\alpha}} + 2\lambda)^\alpha]$$

for  $\lambda \geq 0$ ,  $\delta > 0$  and  $\zeta \geq 0$ . Nevertheless, for  $\alpha \in (-\infty, 0)$ ,  $\psi_{GG}(\infty) = \int_0^\infty \rho_{GG}(dx) < \infty$ , i.e.  $P(T=0) > 0$ , (cfr. also Pitman, 2003, Sec. 5.2), therefore no random probability measure whose ranked atoms induce Gibbs partitions of type  $\alpha < 0$ , can be obtained by normalizing a completely random measure.

**Case b)** For  $\alpha = 0$  a law  $Q_{\gamma, \theta}$  on  $\mathcal{P}_1^\downarrow$  induces an EPPF of Gibbs form (3) if

$$Q_{\theta, \gamma}(\cdot) = \int_0^\infty PD(0, \theta) \gamma(d\theta)$$

where  $PD(0, \theta) = PK(\rho_\theta)$  is the Poisson-Dirichlet ( $\theta$ ) distribution of the ranked atoms of the normalized Gamma process  $P(\cdot) = \frac{\mu(\cdot)}{T}$ , for  $T \sim G(\theta, 1)$  and  $\mu(\cdot) \sim G(\theta H(\cdot), 1)$  as recalled in Example 1. From (4) the EPPF of  $PD(0, \theta)$  arises from letting  $\alpha = 0$  and it is in Gibbs product form. About mixing over  $\theta$  notice that it corresponds to mixing over the total mass of the parameter measure  $\theta H(\cdot)$  of the Dirichlet process, therefore  $Q_{\theta, \gamma}$  provides the distribution of the ranked atoms of *mixtures of Dirichlet processes* as introduced in Antoniak (1974).

It is well-known that this kind of mixing induces correlation (see e.g. Sibisi and Skilling, 1997, Sec. 11) i.e. the resulting random process has no longer independent increments, hence it no longer corresponds to a completely random measure. This implies that, apart from the normalized Gamma process, whose ranked atoms are  $PD(0, \theta)$  distributed, no other normalized random measure can induce exchangeable partitions in Gibbs product form of type  $\alpha = 0$ .

**Case c)** For  $\alpha \in (0, 1)$  a law  $Q_{\alpha, \gamma}$  on  $\mathcal{P}_1^\downarrow$  induces an EPPF of Gibbs form (3) if

$$Q_{\alpha, \gamma}(\cdot) = \int_0^\infty PK(\rho_{\alpha, \delta}|t) \gamma(t) dt \quad (5)$$

for  $\rho_{\alpha, \delta}$ , the Lévy density of the positive  $(\alpha, \delta)$ -stable density  $f_{\alpha, \delta}$

$$\rho_{\alpha, \delta}(x) = \delta 2^\alpha \frac{\alpha}{\Gamma(1-\alpha)} x^{-1-\alpha},$$

and  $\gamma(\cdot)$  an arbitrary probability density on  $(0, \infty)$ . Hence the laws  $Q_{\alpha, \gamma}$  correspond to mixed Poisson-Kingman models  $PK(\rho_\alpha, \gamma)$ , as introduced in Section 2, derived from the positive stable law of index  $\alpha \in (0, 1)$  and scale parameter  $\delta$ . This is the case treated in Lijoi, Prünster and Walker (2007, cfr. Prop. 2). For a general  $\alpha \in (0, 1)$  the conditional structural distribution of  ${}_\alpha \tilde{P}_1|T = t$  which describes  $PK(\rho_{\alpha, \delta}|t)$ , has density

$$\tilde{f}_{\alpha, \delta}(p|t) = \left(\frac{2}{pt}\right)^\alpha \frac{\delta \alpha}{\Gamma(1-\alpha)} \frac{f_{\alpha, \delta}(\bar{p}t)}{f_{\alpha, \delta}(t)} \quad (6)$$

but has no closed form expression since explicit expressions of  $f_{\alpha, \delta}$  are known only in the form of series representation. For  $\alpha = \frac{1}{2}$  the stable density corresponds to the Lévy density,  $f_{\frac{1}{2}, \delta}(t) = \frac{\delta}{\sqrt{2\pi}} t^{-\frac{3}{2}} \exp\{-\frac{\delta^2}{2t}\}$  and by substitution in (6)

$$\tilde{f}_{\frac{1}{2}, \delta}(p|t) = \frac{\delta}{\sqrt{2\pi}\sqrt{pt}} (1-p)^{-\frac{3}{2}} \exp\left\{-\frac{1}{2} \frac{p\delta^2}{(1-p)t}\right\}. \quad (7)$$

For  $\gamma(t) = f_{\alpha,\delta}(t)$ ,  $PK(\rho_{\alpha,\delta}, f_{\alpha,\delta}) = PK(\rho_{\alpha,\delta})$  gives the law of the ranked atoms of a *normalized  $\alpha$ -stable* process, with Laplace exponent  $\psi_{\alpha,\delta}(\lambda) = \delta(2\lambda)^\alpha$ .

Now notice that, in general, apart from the trivial case  $\gamma(t) = f_T(t)$ , *mixed*  $PK(\rho, \gamma)$  models describe the law of the ranked atoms of a random discrete probability measure which can be obtained by normalization of a completely random measure  $P(\cdot)$ , if only if admit an equivalent formulation as *basic* Poisson-Kingman models. The following proposition can be stated that doesn't need to be proved:

**Proposition 1.** Let  $\rho(\cdot)$  be a Lévy density on  $(0, \infty)$ , corresponding to a strictly positive infinitely divisible r.v.  $T$ , let  $\gamma \neq f_T$  be a probability density on  $(0, \infty)$  and  $H$  a fixed distribution on  $(S, \mathcal{S})$  then, a *mixed*  $PK(\rho, \gamma)$  model on  $\mathcal{P}_1^\downarrow$  gives the law of the ranked atoms of a random discrete probability measure  $P(\cdot) = \sum_i P_i \delta_{X_i}(\cdot)$  where  $(X_i)$ , independent of  $(P_i)$ , are iid  $\sim H$ , which admits a construction by normalization of a completely random measure if and only if there exists a Lévy density  $\rho^*(\cdot)$  with  $\int_0^\infty \rho^*(x)dx = \infty$ , such that

$$PK(\rho^*) \equiv PK(\rho, \gamma). \quad (8)$$

Relying on Proposition 1, in case c) our problem reduces to identify the family of mixing distributions  $\gamma(\cdot)$  which satisfy equation (8) for  $\rho = \rho_\alpha$ . To this end, in the following section, we resort to a result stated in Pitman (2003) for which we also provide a proof.

## 4 Pitman's characterization of exponentially tilted PK models.

In Section 4.2 Pitman (2003) focuses on *exponential tilting* as one of the basic operations on Lévy densities which lead to a tractable class of mixed Poisson-Kingman partitions models. The idea of tilting density functions is very old, in Lévy processes setting the equivalent transformation it is also known as *Esscher transform* (see e.g. Sato, 1999). Here we recall the basic definition:

**Definition 1.** [Exponential tilting] Given a probability density  $f$  on  $(0, \infty)$ , with Laplace exponent  $\psi(\lambda)$ , for every  $\lambda > 0$ , the corresponding *exponentially tilted density*  $f_\lambda$  is obtained as

$$f_\lambda(t) = \exp\{-\lambda t + \psi(\lambda) + k(t)\} = \frac{1}{L(\lambda)} e^{-\lambda t} f(t)$$

where  $f(t) = \exp\{k(t)\}$  and  $L(\lambda) = E(e^{-\lambda T}) = \exp\{-\psi(\lambda)\}$ . The corresponding Laplace transform is given by

$$\exp\{-\psi_\lambda(b)\} = \exp\{-\psi(b + \lambda) + \psi(\lambda)\}, \quad (9)$$

for  $b > 0$ , if additionally  $f$  is *infinitely divisible*, then

$$\psi_\lambda(b) = \int_0^\infty (1 - e^{-bs}) e^{-\lambda s} \rho(ds),$$

hence tilting  $f$  yields corresponding tilted Lévy measures,  $\rho_\lambda(\cdot) = e^{-\lambda t} \rho(\cdot)$ .

What Pitman (2003) basically states is that *exponential tilting* is the unique operation on basic PK models that produces mixed PK models satisfying condition of Proposition 1. More exactly

a *basic* model equivalent to a *mixed* model exists if and only if the mixing density belongs to the family of the corresponding tilted densities, in which case the basic model is driven by the tilted version of the Lévy measure of the mixed model. (Notice that the “only if” part of the statement is partially hidden in the last line of Section 4.2 in Pitman’s paper).

Here we recall Pitman’s result, both elucidating the “if” part and providing a proof for the “only if” part.

**Proposition 2.** [Pitman, 2003; Sec. 4.2] Given two regular Lévy densities  $\rho_2$  and  $\rho_1$ , then

$$PK(\rho_2) = PK(\rho_1, \gamma)$$

for some  $\gamma(\cdot)$  if and only if  $\rho_2(t) = \rho_1(t) \exp\{-\lambda t\}$  and  $\gamma(t) = f_1(t) \exp\{\psi_1(\lambda) - \lambda t\}$ , for every  $t$ , for some  $\lambda > 0$ , where  $f_1(\cdot)$  is the probability density corresponding to  $\rho_1$  and  $\psi_1(\lambda)$  its Laplace exponent.

*Proof (Sufficiency)* Let  $\rho(\cdot)$  be a Lévy density corresponding to a probability density  $f(\cdot)$ , and  $\rho_\lambda(\cdot) = e^{-\lambda t} \rho(\cdot)$  the tilted Lévy density of the corresponding tilted probability density  $f_\lambda(\cdot) = f_1(t) \exp\{\psi_1(\lambda) - \lambda t\}$ . It is easy to show that the conditional structural distributions for the models  $PK(\rho|t)$  and  $PK(\rho_\lambda|t)$  coincide, in fact

$$\tilde{f}_\lambda(p|t) = pt\rho_\lambda(pt) \frac{f_\lambda(\bar{p}t)}{f_\lambda(t)} = pte^{-\lambda pt} \rho(pt) \frac{e^{\psi(\lambda) - \lambda \bar{p}t} f(\bar{p}t)}{e^{\psi(\lambda) - \lambda t} f(t)} = pt\rho(pt) \frac{f(\bar{p}t)}{f(t)} = \tilde{f}(p|t).$$

It follows that  $PK(\rho_\lambda|t) = PK(\rho|t)$  for every  $t$ , hence

$$PK(\rho_\lambda, \gamma) = PK(\rho, \gamma)$$

for every mixing density  $\gamma$ . For  $\gamma_\lambda = f_\lambda = f_1(t) \exp\{\psi_1(\lambda) - \lambda t\}$ , this implies  $PK(\rho_\lambda, \gamma_\lambda) = PK(\rho, \gamma_\lambda)$ , hence

$$PK(\rho_\lambda) = PK(\rho_\lambda, \gamma_\lambda) = PK(\rho, \gamma_\lambda).$$

□

*(Necessity)* We want to show that if  $\rho_1$  is the Lévy density of some infinitely divisible density  $f_1(\cdot)$  and there exists a Lévy density  $\rho_2(x) \neq \rho_1(x)$  such that, for some density  $\gamma$

$$PK(\rho_2) = PK(\rho_1, \gamma), \tag{10}$$

then  $\rho_2(\cdot) = e^{-\lambda t} \rho_1(\cdot)$  and  $\gamma(t) = e^{\psi_1(\lambda) - \lambda t} f_1(t)$ , for  $\lambda > 0$  and  $\psi_1(\cdot)$  the Laplace exponent of  $f_1(t)$ . Now, call  $f_2$  the probability density corresponding to the Lévy density  $\rho_2$  of some infinitely divisible r.v.  $T_2$ . Since  $PK(\rho_2) = PK(\rho_2, f_2)$ , then (10) equates

$$PK(\rho_2, f_2) = PK(\rho_1, \gamma). \tag{11}$$

Condition (11) implies that all the following three conditions must hold:

- a) there exists some function  $\phi(\cdot)$  such that  $\gamma(t) = \phi(t)f_1(t) = f_2(t)$  for every  $t$ ,
- b) there exists a function  $g(\cdot)$  on  $(0, \infty)$  such that, for  $b > 0$

$$E(e^{-\lambda T_2}) = \int_0^\infty e^{-bt} \phi(t) f_1(t) dt = \exp\{-\psi_2(b)\} = \exp - \int_0^\infty (1 - e^{-bx}) g(x) \rho_1(dx)$$

is the Laplace transform of  $T_2$ ,



c)  $PK(\rho_2|t) = PK(\rho_1|t)$ , for every  $t$ .

Condition b) can be rewritten as

$$\int_0^\infty e^{\psi_2(b)} e^{-bt} \phi(t) f_1(t) dt = 1$$

for which a family of possible solutions for every  $b > 0$  is given by:

$$\phi(t, b) = e^{-\psi_2(b)+bt}. \quad (12)$$

Conditions b) and c) imply

$$\rho_1(pt) \frac{f_1(\bar{p}t)}{f_1(t)} = \rho_1(pt) g(pt) \frac{f_1(\bar{p}t) \phi(\bar{p}t)}{f_1(t) \phi(t)}$$

for all  $t$  and for all  $0 < p < 1$ , which yields

$$\phi(t) = \phi(\bar{p}t) g(pt). \quad (13)$$

For the uniqueness of the Laplace transform, the density  $f_2(t)$  corresponding to  $\psi_2$  must be unique, therefore combining conditions (12) and (13) one obtains

$$e^{-\psi_2(b)+bt} = e^{-\psi_2(b)+b\bar{p}t} g(pt)$$

which yields  $g(pt) = e^{bpt}$  for  $b > 0$ . It follows that the transformed Lévy density must be of the form  $\rho_2 = e^{bt} \rho_1$ . By Example 33.15 in Sato (1999), for  $b < 0$ ,  $g(x) = e^{bx}$  is the *exponential transformation*, hence by the change of variable  $\lambda = -b$ , it follows that  $\rho_2 = e^{-\lambda t} \rho_1(\cdot)$  and the corresponding tilted probability density

$$f_2(t) = e^{\psi_1(\lambda)-\lambda t} f_1(t)$$

are the unique solutions to (10). □

**Remark 1.** [Dirichlet process] By Example 1,  $PK(\rho_\theta) = PK(\rho_\theta, \gamma)$  for every  $\gamma$ , hence  $PK(\rho_\theta) = PK(\rho_\theta, \gamma_{\theta, \lambda})$  for  $\gamma_{\theta, \lambda}$  the tilted Gamma density. Moreover, by Proposition 2,  $PK(\rho_{\theta, \lambda}) = PK(\rho_\theta, \gamma_{\theta, \lambda})$ , for  $\rho_{\theta, \lambda}$  the corresponding Lévy measure, hence  $PK(\rho_\theta) = PK(\rho_{\theta, \lambda})$  for every  $\lambda$ .

It is easy in fact to verify that the operation of tilting a Gamma  $(\theta, 1)$  density

$$f_\lambda(t) = \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-x} \exp\{-\theta \log(1+\lambda) - \lambda x\} = \frac{(1+\lambda)^\theta}{\Gamma(\theta)} x^{\theta-1} e^{-x(1+\lambda)}$$

yields a family of Gamma $(\theta, (1+\lambda))$  laws.

We are now in a position to complete the discussion of case c) of Theorem 1. By exploiting Proposition 2, the solution to case c) arises immediately. The law  $Q_{\alpha, \gamma}(\cdot)$  in (5) admits a representation as a basic PK models, if and only if  $\gamma$  belongs to the family of the *exponentially tilted positive stable* densities, (first introduced in Hougaard, 1986, also called *tempered stable* distributions in Barndorff-Nielsen and Shepard, 2001) given by,

$$f_{\alpha, \delta}^\lambda(t) = f_{\alpha, \delta}(t) \exp\{\delta \lambda^\alpha - \lambda t\}.$$

The basic model is driven by the corresponding tilted Lévy density

$$\rho_{\alpha, \delta}^\lambda(s) = \frac{\delta 2^\alpha \alpha}{\Gamma(1-\alpha)} s^{-1-\alpha} e^{-\lambda s}.$$

By the change of variable  $\lambda = \frac{\zeta^\frac{1}{\alpha}}{2}$ , and applying (9), the Laplace exponent of  $f_{\alpha,\delta}^\lambda$  results

$$\psi_{\alpha,\delta}^\zeta(b) = -\delta\zeta + \delta(\zeta^\frac{1}{\alpha} + 2b)^\alpha \quad (14)$$

which is well-known to identify to the family of infinitely divisible *generalized Gamma distributions* (Brix, 1999,) as already recalled in the discussion of case a). Notice that, contrary to case  $\alpha < 0$ , for  $\alpha \in (0, 1)$ ,  $\psi_{GG}(\infty) = \infty$ , hence  $P(T > 0) = 1$ .

**Example 2.** [Inverse-Gaussian process] As previously recalled for  $\alpha = \frac{1}{2}$  the positive *stable* density has explicit expression

$$f_{\frac{1}{2},\delta}(t) = \frac{\delta}{\sqrt{2\pi}} t^{-\frac{3}{2}} e^{-\frac{\delta^2}{2t}},$$

with corresponding Laplace exponent  $\psi(\lambda) = \delta\sqrt{2\lambda}$  and Lévy density  $\rho_{\frac{1}{2},\delta} = \frac{\delta}{\sqrt{2\pi}} x^{-\frac{3}{2}}$ . By exponential tilting with  $\lambda = \frac{\zeta^2}{2}$ , the *exponentially tilted*  $1/2$ -stable density results

$$f_{\delta,\zeta}(t) = \frac{\delta}{\sqrt{2\pi}} e^{\delta\zeta} t^{-\frac{3}{2}} \exp\left\{-\frac{1}{2}(\delta^2 t^{-1} + \zeta^2 t)\right\}, \quad (15)$$

for  $\delta \in (0, \infty)$  and  $\zeta \in [0, \infty)$ , which is well-known to be the density of an *inverse Gaussian*  $(\delta, \zeta)$  law (see e.g. Seshadri, 1993). By (14) corresponding Laplace exponent results  $\psi_{\frac{1}{2},\delta}^\zeta(b) = -\delta\zeta + \delta(\zeta^2 + 2b)^\frac{1}{2}$  and the corresponding tilted Lévy density is given by:

$$\rho_{\frac{1}{2},\delta}^\zeta(x) = \frac{\delta}{\sqrt{2\pi}} x^{-\frac{3}{2}} e^{-\frac{\zeta^2}{2} x^2}.$$

By Proposition 2

$$PK(\rho_{\frac{1}{2},\delta}^\zeta) = PK(\rho_{\frac{1}{2},\delta}, f_{\delta,\zeta}),$$

and it is an easy task to verify that the conditional structural distribution derived from the inverse Gaussian density equates (7). Applying (2) for  $f_{\alpha,\zeta}$  as in (15)

$$\tilde{f}_{\delta,\zeta}(p|t) = \frac{pt\delta}{\sqrt{2\pi}} (pt)^{-\frac{3}{2}} \exp\left\{-\frac{pt}{2}\zeta^2\right\} \frac{(pt)^{-\frac{3}{2}} \exp\left\{-\frac{1}{2}(\delta^2((1-p)t)^{-1} + \zeta^2(1-p)t)\right\}}{t^{-\frac{3}{2}} \exp\left\{-\frac{1}{2}(\delta^2 t^{-1} + \zeta^2 t)\right\}},$$

and some elementary calculations show it simplifies to

$$\tilde{f}_{IG}(p|t) = \frac{\delta}{\sqrt{2\pi}\sqrt{pt}} (1-p)^{-\frac{3}{2}} \exp\left\{-\frac{1}{2} \frac{p\delta^2}{(1-p)t}\right\}.$$

Notice that the corresponding normalized random measure, termed *normalized Inverse Gaussian process* has been derived by mimicking Ferguson's (1973) construction of the Dirichlet process, and studied in relation to hierarchical Bayesian nonparametric mixture modeling in Lijoi, Mena and Prünster (2005).

This complete the discussion of case a), b), and c) of Gneden and Pitman's characterization of Gibbs partitions. We have shown that for  $\alpha \in (-\infty, 0)$  nor the extreme partitions, nor the family of mixture partition models, contain elements that can be derived by means of normalization of completely random measures. For  $\alpha = 0$ , only the family of extreme partitions  $\{PD(0, \theta), \theta > 0\}$ , correspond to the ranked atoms of a NRM, namely the *normalized Gamma* process, which belongs

to the family of Generalized Gamma processes for  $\alpha = 0$ . Finally for  $\alpha \in (0, 1)$  only mixing exponentially tilted stable distributions of index  $\alpha \in (0, 1)$  produce partitions models for random discrete probability measures that can be derived by normalization of completely random measures, namely *generalized Gamma* processes of index  $\alpha \in (0, 1)$ .

We summarize our conclusions in the following proposition, slightly more general than that given in Lijoi, Prünster and Walker (2007, Prop. 2).

**Proposition 3.** The unique family of random discrete probability measures which admits a construction through normalization of completely random measures, and induces exchangeable partition probability functions in Gibbs product form of type  $\alpha < 1$  is the class of normalized generalized Gamma processes of index  $\alpha \in [0, 1)$ .

In Cerquetti (2007) an explicit form of the EPPF of exponentially tilted Poisson-Kingman models derived from the positive stable subordinator for  $\alpha \in (0, 1)$ , that here we recall for completeness, is derived from the general form for mixed PK models given in Pitman (2003), (see also Lijoi, Mena and Prünster, 2007),

$$p_{GG}(n_1, \dots, n_k) = \frac{e^{\delta\gamma} \delta^k \alpha^k 2^n}{\Gamma(n)} \prod_{j=1}^k (1 - \alpha)_{n_j - 1} \int_0^\infty \lambda^{n-1} \frac{e^{-\delta(\gamma^{\frac{1}{\alpha}} + 2\lambda)^\alpha}}{(\gamma^{\frac{1}{\alpha}} + 2\lambda)^{n-k\alpha}} d\lambda.$$

Notice the Gibbs product form of type  $\alpha$  of (16), where

$$V_{n,k} = \frac{e^{\delta\gamma} \delta^k \alpha^k 2^n}{\Gamma(n)} \int_0^\infty \lambda^{n-1} \frac{e^{-\delta(\gamma^{\frac{1}{\alpha}} + 2\lambda)^\alpha}}{(\gamma^{\frac{1}{\alpha}} + 2\lambda)^{n-k\alpha}} d\lambda.$$

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